# Voting rules as Group Decision Making Models

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Chapter 3

## Aims

- Study decision problems in which a group has to take a decision among several alternatives
- Analyze a number of properties of electoral systems
- Present a few elements of the classical theory: Social choice theory
- Parameters to take into account:
  - nature of the decision
  - size of the group
  - nature of the group
- Many (deep) results
  - Economics, Political Science, Applied Mathematics, Operation Research
  - Two Nobel Prizes: Kenneth J. Arrow, Amartya Sen

### Problem

Study election problems in which a society has to take a decision among several candidates

## Election of one candidate

- Common sense:
  - the choice of the candidate will affect all members of the society
  - the choice of the candidate should take the opinion of all members of society into account
- Intuition:
  - Democracy $\Longrightarrow$ Elections $\Longrightarrow$ Majority

## Political problems

- direct or indirect democracy?
- rôle of parties?
- who can vote? (age, sex, nationality, paying taxes,. . . )
- who can be candidate?
- what type of mandate?
- how to organize the campaign?
- rôle of polls?

### Technical problems

- Majority: When there are only two candidates
  - elect the one receiving the more votes
- Majority: When there are more than two candidates
  - many ways to extend this simple idea
  - not equivalent
  - sometimes leading to unwanted results

## Typology of elections

- Two main criteria
  1. type of ballots admitted
     - one name
     - ranking of all candidates
     - other types (acceptable candidates, grading candidates, etc.)
  2. method for organizing the election and for tallying ballots
- Consequences:
  - many possible types of elections
  - many have been proposed
  - many have have been used in practice

## Two hypotheses

1. All voters are able to rank order the set of all candidates (ties admitted)
   - e.g. each voter has a weak order on the set of all candidates:

$$a \succ b \succ c \sim d \succ e$$

2. Voters are sincere
   - if I have to vote for one candidate, I vote for a

# *Plurality voting*

### Rules

- one round of voting
- ballots with one name
- “first past the post”

### Remark

- ties are neglected (unlikely)
  - one voter has special power (the Queen chooses in case of a tie)
  - one candidate receives special treatment (the older candidate is elected)
  - random tie breaking rule

# *Plurality voting*

## Example

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Which candidate is elected ?

# *Plurality voting*

## Remarks

- Problems are expected as soon as there are more than 2 candidates
- A system based on an idea of “majority” may well violate the will of a majority of voters
- Sincerity hypothesis is heroic!

# *Plurality with runoff*

## Rules

- Ballots with one name
- First round
  - the candidate with most votes is elected if he receives more than 50% of votes
  - otherwise go to the second round
- Second round
  - keep the two candidates having received more votes
  - apply plurality voting

# *Plurality with runoff*

Example (Previous Example)

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Which candidate is elected ?

# *Plurality with runoff*

**Example**

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

Which candidate is elected ?

# *Plurality with runoff*

### Plurality vs plurality with runoff

- The French system does only a little better than the UK one
- Preferences used in the above example are not bizarre.

# *Plurality with runoff : manipulation*

## Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

$b$ is elected

## Non-sincere voting

- Suppose that the 6 voters for which $c \succ a \succ d \succ b$ vote as if their preferences were $a \succ c \succ d \succ b$
- Result:
  - $a$ is elected at the first round (11/21)
  - profitable to the six manipulating voters (for them $a \succ b$)

# *Manipulable voting rules*

### Definition

- A voting rule is manipulable if it may happen that some voters may have an interest to vote in a non-sincere way

### Remarks

- Plurality with runoff is manipulable

# *Plurality with runoff: monotonicity*

## Before campaign

- 3 candidates $\{a, b, c\}$
- 17 voters

$$\begin{array}{ll} 6 \text{ voters:} & a \succ b \succ c \\ 5 \text{ voters:} & c \succ a \succ b \\ 4 \text{ voters:} & b \succ c \succ a \\ 2 \text{ voters:} & b \succ a \succ c \end{array}$$

Which candidate is elected?

# Plurality with runoff: monotonicity

## Before campaign

- 3 candidates $\{a, b, c\}$
- 17 voters

$$\begin{array}{ll} 6 \text{ voters:} & a \succ b \succ c \\ 5 \text{ voters:} & c \succ a \succ b \\ 4 \text{ voters:} & b \succ c \succ a \\ 2 \text{ voters:} & b \succ a \succ c \end{array}$$

- Suppose that last 2 voters ($b \succ a \succ c$) change their minds in favor of $a$
- Their new preferences are $a \succ b \succ c$

Which candidate is elected?

# *Condorcet voting rule (1785)*

## Principles

- compare all candidates by pair
- declare that $a$ is "socially preferred" to $b$ if (strictly) more voters prefer $a$ to $b$ (social indifference in case of a tie)
- Condorcet's principle: if one candidate is preferred to all other candidates, it should be elected. This candidate is called a Condorcet Winner
- Condorcet Winner (CW: must be unique)

## Remarks

- Plurality rule and Plurality with runoff violate Condorcet's principle
- Condorcet's principle does not solve the "dictature of the majority" difficulty
- a Condorcet winner is not necessarily "ranked high" by voters

# *Condorcet voting rule*

Example

- 3 candidates $\{a, b, c\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & a \succ b \succ c \\ 6 \text{ voters:} & b \succ c \succ a \\ 5 \text{ voters:} & c \succ b \succ a \end{array}$$

Is there a Condorcet winner?

# *Condorcet voting rule*

Example

- 4 candidates $\{a, b, c, d\}$
- 21 voters (or 21 000 000 or 42 000 000, ...)

$$\begin{array}{rl} 10 \text{ voters:} & b \succ a \succ c \succ d \\ 6 \text{ voters:} & c \succ a \succ d \succ b \\ 5 \text{ voters:} & a \succ d \succ b \succ c \end{array}$$

Is there a Condorcet winner?

# *Condorcet's Paradox*

## Example

- 3 candidates $\{a, b, c\}$
- 3 voters

$$\begin{array}{ll} 1 \text{ voters:} & a \succ b \succ c \\ 1 \text{ voters:} & b \succ c \succ a \\ 1 \text{ voters:} & c \succ a \succ b \end{array}$$

the social strict preference relation may have circuits!

## Electing the CW

- attractive but not always effective!

# *Borda voting rule (1783)*

## Principles

- Each ballot is an ordered list of candidates (exclude ties for simplicity)
- On each ballot compute the rank of the candidates in the list
- Rank order the candidates according to the decreasing sum of their ranks

## Remarks

- simple
- efficient: always lead to a result
- separable, monotonic, participation incentive

# *Borda voting rule*

Example

- 4 candidates $\{a, b, c, d\}$
- 3 voters

$$\begin{array}{ll} 2 \text{ voters:} & b \succ a \succ c \succ d \\ 1 \text{ voters:} & a \succ c \succ d \succ b \end{array}$$

Which candidate is elected by using a Borda procedure?

# Summary

## Example

- 4 candidates $\{a, b, c, d\}$
- 27 voters (may be also 27 000 000 or 54 000 000, . . . )

$$\begin{array}{ll} 5 \text{ votants :} & a \succ b \succ c \succ d \\ 4 \text{ votants :} & a \succ c \succ b \succ d \\ 2 \text{ votants :} & d \succ b \succ a \succ c \\ 6 \text{ votants :} & d \succ b \succ c \succ a \\ 8 \text{ votants :} & c \succ b \succ a \succ d \\ 2 \text{ votants :} & d \succ c \succ b \succ a \end{array}$$

Determine the candidate elected by using the plurality, plurality with runoff, Condorcet principle and Borda principle.

# *What are we looking for?*

**Democratic method**

- always giving a result like Borda
- always electing the Condorcet winner
- consistent w.r.t. withdrawals
- monotonic, separable, incentive to participate, not manipulable etc.

# *Arrow*

## Framework

- $n \geq 3$ candidates (otherwise use plurality)
- $m$ voters ($m \geq 2$ and finite)
- ballots: ordered list of candidates
- A voting profile is denoted by $(\succsim_i)_{i=1,\dots,m}$ where $\succsim_i$ is an individual preferences of the voter $i$.
- The result (collective preference) of the voting is denoted by $\succsim$.

## Problem

- find all electoral methods respecting a small number of "desirable" principles

# *Arrow*

## Principles

- Universality
  - the method should be able to deal with any configuration of ordered lists, i.e, there is no restriction about the expression of a voter.
- Transitivity
  - the result of the method should be an ordered list of candidates
- Unanimity
  - the method should respect a unanimous preference of the voters

$$\forall x, y, \ [x \succsim_i y \quad \forall i = 1, \ldots, m] \Longrightarrow x \succsim y$$

# Arrow

## Principles

- Absence of dictator
  - the method should not allow for dictators

$$\exists i_0, \forall x, y \; [x \succsim_{i_0} y \Longrightarrow x \succsim y]$$

- Independence of irrelevent alternatives
  - the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters

$$\forall x, y, \; [\forall i = 1, \ldots, m, \; x \succsim_i y \Longleftrightarrow x \succsim_i^{'} y] \Longrightarrow [x \succsim y \Longleftrightarrow x \succsim^{'} y]$$

# *Arrow's theorem (1951)*

**Theorem**

There is no method respecting the five principles

**Borda**

- universal, transitive, unanimous with no dictator
- cannot be independent

**Condorcet**

- universal, independent, unanimous with no dictator
- cannot be transitive

## Exercise

We consider the following profile (9 voters and 4 candidates) where the preferences of the last voter are unknown:

$$\begin{array}{rl} 4 \text{ voters:} & c \succ d \succ a \succ b \\ 2 \text{ voters:} & a \succ b \succ d \succ c \\ 2 \text{ voters:} & b \succ a \succ c \succ d \\ 1 \text{ voter:} & ? \succ ? \succ ? \succ ? \end{array}$$

1. Do we necessarily know the preferences of the last voter, in order to determine the result of the elections in a UK system (plurality) and French system (plurality with runoff)? If yes, gives these preferences and the results of these elections.
2. Does the Condorcet winner exist in this election?
3. Which preferences the last voter should have in order to elect $a$ as the Condorcet winner? Same question with $b$, $c$ or $d$.